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TRANSLATION-INVARIANT SUBSPACES OF FUNCTIONS ON THE GROUP OF LINEAR TRANSFORMATIONS ON THE REAL LINE[†]

BY
YITZHAK WEIT

ABSTRACT

The characterization of right translation-invariant subspaces of $L_\infty(G^*)$, where

$$G^* = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbf{R}^+, b \in \mathbf{R} \right\},$$

is studied. We introduce the class of multiplier functions which, in the semisimple case, play a role similar to that played by the exponentials for the real line. However, it is proved that multiplier functions of G^* with respect to $\mathbf{R}$ fail to characterize right translation-invariant subspaces of $L_\infty(G^*)$. That is, we construct a right translation-invariant, w*-closed subspace of $L_\infty(G^*)$ which contains no multiplier function.

1. Introduction

Wiener's Theorem in spectral analysis of bounded functions on the real line states that every non-trivial, translation-invariant, w*-closed subspace of $L_\infty(\mathbf{R})$ contains an exponential function. Segal and Godement generalized Wiener's result to arbitrary abelian locally compact groups [6, 10]. In the non-abelian case, the two-sided analogue of Wiener's Theorem may be formulated as follows: Every non-trivial, two-sided translation-invariant, w*-closed subspace of $L_\infty(G)$ contains an indecomposable positive definite function. In [7] it was proved that the two-sided analogue of Wiener's Theorem holds for all connected nilpotent Lie groups, and for all semi-direct products of abelian groups. It is known that Wiener's Theorem does not hold for any non-compact connected semisimple Lie group [2, 7].

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We are going to study the problems of one-sided spectral analysis of bounded functions on locally compact groups. Our purpose is to find a class of bounded functions which will play a role similar to that played by the exponentials for the real line.

Actually, for a non-abelian group G we may pose the following problems:

(i) Find a class F (as small as possible), $F \subset L_\infty(G)$, such that every right invariant, w*-closed subspace of $L_\infty(G)$ contains a member of F.

(ii) Determine the fundamental functions in F, i.e., functions which generates minimal, non-trivial, right invariant, w*-closed subspaces.

(iii) Does every non-trivial, right invariant, w*-closed subspace of $L_\infty(G)$ contain a fundamental function of F?

For the two-dimensional Euclidean motion group these three problems are solved in [11], [13].

Let G^* denote the group of linear transformations on the real line of the form $ax+b$, $b \in \mathbf{R}$, $a \in \mathbf{R}^+$ ($\mathbf{R}^+$ denotes the multiplicative group of positive reals). G^* may be identified with the group

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbf{R}^+, b \in \mathbf{R} \right\}.$$

The problems of two-sided spectral analysis for G^* are easily solved in [12, 9] where the class F is provided by the characters of G^*. However, the situation is completely different for the one-sided problem: the subspace $\{\Phi(a)e^{ib} : \Phi \in L_\infty(\mathbf{R})\}$ is a right invariant, w*-closed subspace of $L_\infty(G^*)$ which contains no indecomposable positive definite function.

To generalize the notion of an exponential function to an arbitrary group, regarding one-sided spectral analysis, we follow [5].

2. Multiplier functions

If G is a topological group, by a G-space is meant a space X and a continuous map $(g,x) \to gx$ of $G \times X \to X$ satisfying $(g_1g_2)x = g_1(g_2x)$ and $ex = x$ for e the identity of G. X is a homogeneous G-space if G is transitive on X.

The exponentials are characterized by the property that they have unit value at the identity and any translate is proportional to the function itself. This condition is modified by supposing instead that we have a family of functions $s(\cdot,\xi)$, where ξ ranges over some index space, with the property that $s(e,\xi) = 1$ and that a right translate of each of the $s(\cdot,\xi)$ is proportional to some other of the $s(\cdot,\xi)$. It may be shown that the following functional equation is satisfied:

$$s(gg', \xi) = s(g, g'\xi)s(g', \xi) \tag{2.1}$$

where ξ ranges over a G-space X.

In [5] a positive continuous function s which satisfies (2.1) is referred to as a multiplier and the function $s(\cdot, \xi)$ is called a multiplier function.

If X consists of a single point then (2.1) defines a positive character on G. Thus the positive exponentials are the multiplier functions on $\mathbf{R}$ corresponding to a trivial X.

It is shown in [5] that in the case of a semi-simple G, there is a choice of X for which the multiplier functions play the role of the exponentials.

Let μ be a positive bounded Borel measure on $\mathbf{R}^n$ that does not have its support in a proper subgroup of $\mathbf{R}^n$. Let V denote the cone of non-negative measurable functions f on $\mathbf{R}^n$ satisfying $\mu * f = \lambda f$ where $\lambda > 0$.

A theorem of Choquet and Deny [1] states that each member of V has a unique integral representation in terms of extremal functions which turn out to be the positive exponentials in V. The following generalization of the Choquet–Deny Theorem for a semi-simple Lie group G was announced in [5]:

Let μ be a positive absolutely continuous measure whose compact support contains a neighbourhood of the identity. Let $V_\lambda(\mu)$ denote the cone of nonnegative solutions to

$$\int f(g'g)d\mu(g') = \lambda f(g), \qquad \lambda > 0. \tag{2.2}$$

Then every member of $V_\lambda(\mu)$ admits a Choquet–Deny representation in terms of its normalized extremals. Each normalized extremal is a multiplier function with respect to the homogeneous space $B(G)$ called the boundary of G.

Thus, in the problem of Choquet and Deny the multiplier functions of G with respect to $B(G)$ play the role of the exponentials.

Positive exponentials generates irreducible translation-invariant cones and in fact this property characterizes the exponentials [5]. It was verified in [5] that in the semi-simple case the right translation-invariant irreducible cones satisfying an additional condition are generated by multiplier functions.

Another example in which right invariant subspaces of functions satisfying convolution equations are spanned by their multiplier functions, appears in the Poisson formula for harmonic functions on a semi-simple group G [4].

Let μ be a probability measure on G which is absolutely continuous and whose support contains a neighbourhood of the identity of G. The bounded solutions to (2.2) for $\lambda = 1$ are called μ-harmonic functions. It was proved by H. Furstenberg [4] that given μ, there is determined a probability measure γ on the

G-space $B(G)$ with the property that every μ-harmonic function f is given by:

$$f(g) = \int_{B(G)} \tilde{f}(g^{-1}x)d\gamma(x) \tag{2.3}$$

where $\tilde{f}$ is a bounded measurable function on $B(G)$. As is shown in [5] we may write (2.3) as

$$f(g) = \int_{B(G)} \frac{dg^{-1}\gamma}{d\gamma}(x)\tilde{f}(x)d\gamma(x)$$

and

$$s(g,x) = \frac{dg^{-1}\gamma}{d\gamma}(x) \quad \text{is a multiplier with respect to } B(G).$$

Thus, the measure μ on G determines a multiplier s with respect to $B(G)$ such that each μ-harmonic function f is an integral over multiplier functions corresponding to s.

A similar Poisson formula was obtained for the group G^* where the homogeneous space $G^*/\mathbf{R}^+ \simeq \mathbf{R}$ is a boundary of G^* [3]. That is, every μ-harmonic function on G^* is an integral over multiplier functions with respect to $\mathbf{R}$.

These results suggest that multiplier functions of G^* with respect to $\mathbf{R}$ may play the role of the exponentials in one-sided spectral analysis.

For our usage we take the following as our definition.

Definition. Let X be a G-space. A bounded, measurable, non-zero function $s(g,\xi)$ on $G \times X$ which satisfies (2.1) is called a multiplier. The function $s(\cdot,\xi)$ is called a multiplier function.

Let $G = N \rtimes H$ denote the semi-direct product of the groups N and H and let $h \to \tau_h$ be the homomorphism which carries H onto a group of automorphisms of N. For a locally compact abelian group G, let $\hat{G}$ denote the character group of G and e_G the identity element of G.

For semi-direct products we have the following characterization of multiplier functions:

Proposition. *Let $G = N \rtimes H$ where N and H are locally compact abelian groups. Then each multiplier s of G with respect to the G-space $G/H \simeq N$ is of the form*

$$s(g,x) = \chi(h)\frac{\phi(gx)}{\phi(x)}$$

where $g=(n,h)\in G$, $\chi\in\hat{H}$ and ϕ is a bounded measurable function which does not vanish in N.

PROOF. Let $s((n,h),x)$ be a multiplier of G with respect to the G-space N. Consequently, s satisfies the following:

$$s((n_1,\tau_{h_1}(n_2),h_1h_2),x)=s((n_1,h_1),n_2\tau_{h_2}(x))s((n_2,h_2),x) \tag{2.4}$$

for every n_1, n_2, $x\in N$ and $h_1,h_2\in H$, and

$$s(e_G,x)=1 \qquad \text{for every } x\in N. \tag{2.5}$$

Let $f(n,h)=s((n,h),e_N)$. If $f(n_0,e_H)=0$ for some $n_0\in N$ then from (2.4) we deduce

$$s((e_N,e_H),e_N)=s((n_0^{-1},e_H),e_N)s((n_0,e_H),e_N)=0$$

in contradiction to (2.5). Thus, we may assume that $f(n,e_H)$ does not vanish in N. From (2.4) we have

$$s((n,h),x)=\frac{f(n\tau_h(x),h)}{f(x,e_H)} \tag{2.6}$$

and $f(n,h)=f(n,e_H)f(e_N,h)$ for $n\in N$ and $h\in H$.

The function $\chi(h)=f(e_N,h)$ is a bounded measurable function on H which satisfies

$$\chi(h_1h_2)=\chi(h_1)\chi(h_2) \qquad \text{for every } h_1,h_2\in H \quad \text{and} \quad \chi(e_H)=1.$$

Hence $\chi\in\hat{H}$. Let $\phi(x)=f(x,e_H)$. Then (2.6) may be written as

$$s((n,h),x)=\chi(h)\frac{\phi(n\tau_h(x))}{\phi(x)}$$

and the result follows.

For $G^*=\mathbf{R}\rtimes\mathbf{R}^+$ the multipliers with respect to the G^*-space $\mathbf{R}$ are of the form

$$s(g,x)=\frac{a^{i\theta}\phi(ax+b)}{\phi(x)} \qquad \text{where } g=\begin{pmatrix} a & b\\ 0 & 1\end{pmatrix}, \quad b,\theta\in\mathbf{R}, \quad a\in\mathbf{R}^+$$

and ϕ is a bounded measurable function that does not vanish on $\mathbf{R}$.

Our multiplier conjecture regarding one-sided spectral analysis for $L_\infty(G^*)$ may be formulated as follows:

Every right translation invariant, w*-closed, non-trivial subspace of $L_\infty(G^*)$ contains a function of the form $a^{i\theta}\phi(b)$ where $\phi\in L_\infty(\mathbf{R})$, $\phi\neq 0$ and $\theta\in\mathbf{R}$.

3. A counterexample

Let G denote the group $\{(x,y): x \in \mathbf{R},\ y \in \mathbf{R}^+\}$ where $(x,y)\circ(x',y') = (xy'+x', yy')$. With the topology induced by the upper half-plane, G is a locally compact group isomorphic to G^*, as seen by the map

$$i: \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \to \left(\frac{b}{a}, \frac{1}{a}\right) \quad \text{from } G^* \text{ on } G.$$

The multiplier conjecture for G turns out to be the following: every right invariant, w*-closed, non trivial subspace of $L_\infty(G)$ contains a function of the form $y^{i\theta}\phi(x/y)$ where $\phi \in L_\infty(\mathbf{R})$, $\phi \neq 0$ and $\theta \in \mathbf{R}$.

Let $d\mu = (1/y^2)dydx$ denote the right Haar measure on G.

For $f \in L_\infty(G)$ let N_f denote the w*-closed subspace generated by right translates of f. Let $\mathrm{Sp}(\phi)$ denote the spectrum of $\phi \in L_\infty(\mathbf{R})$. Let $S(\mathbf{R})$ denote the space of functions f in $C^\infty(\mathbf{R})$ which satisfy

$$\lim_{|x|\to\infty} x^n |f^{(j)}(x)| = 0 \qquad \text{for } n = 0,1,2,\cdots, \quad j = 0,1,2,\cdots.$$

A counterexample to the multiplier conjecture is given by the following theorem.

THEOREM. *Let $\Phi(t)$ be a piece-wise linear function defined for $t > 0$, which satisfies the following condition:*

For each real λ, $\varepsilon > 0$ and $M > 0$ there exist real numbers t_1, t_2 and λ_0 such that $t_2 - t_1 > M$, $|\lambda_0 - \lambda| < \varepsilon$ and $\Phi'(t) = \lambda_0$ for $t_1 < t < t_2$.

Let $f(x,y) = g(y)e^{-ix}$ where

$$g(y) = \begin{cases} e^{i\Phi(\ln y)} & y > 1, \\ 0 & 0 < y \leqq 1. \end{cases}$$

Then the w*-*closed subspace $N_f \subset L_\infty(G)$ generated by right translates of f contains no function of the form $y^{i\theta}\phi(x/y)$ where $\phi \in L_\infty(\mathbf{R})$, $\phi \neq 0$ and $\theta \in \mathbf{R}$.*

The proof of the theorem will be accomplished in several lemmas.

Let $\Phi(t) = \lambda t + b_\lambda$, $b_\lambda \in \mathbf{R}$, for $\underline{t}_\lambda < t \leqq \bar{t}_\lambda$ where the intervals $(\underline{t}_\lambda, \bar{t}_\lambda]$ satisfy the condition stated in the theorem. Let $\underline{y}_\lambda = e^{\underline{t}_\lambda}$, $\bar{y}_\lambda = e^{\bar{t}_\lambda}$ and $d_\lambda = e^{ib_\lambda}$. Then g may be described as

$$g(y) = \begin{cases} d_\lambda y^{i\lambda} & \underline{y}_\lambda < y \leqq \bar{y}_\lambda, \\ 0 & 0 < y \leqq 1. \end{cases}$$

For $0 < \varepsilon < 1$ let

$$B_{\lambda,\varepsilon} = \left(\frac{1}{1-\varepsilon}\underline{y}_\lambda, \frac{1}{1+\varepsilon}\bar{y}_\lambda\right) \qquad \text{whenever } \frac{1}{1-\varepsilon}\underline{y}_\lambda < \frac{1}{1+\varepsilon}\bar{y}_\lambda.$$

Let Γ denote the set of the real numbers λ that appear in the definition of g.

REMARK. For $f(x, y) = g(y)e^{-ix}$ the subspace N_f is the w*-closed subspace spanned by $\{g(\mu y)e^{-i\mu x} : \mu > 0\}$.

We prove the following:

LEMMA 3.1. *Let $\psi \in N_f$. Then for every $G \in S(\mathbf{R})$ with $\operatorname{Supp} \hat{G} \subset [1-\varepsilon, 1+\varepsilon]$, $0 < \varepsilon < 1$ and for every $\lambda \in \Gamma$, there exists a constant $C^*(\lambda; G)$ such that*

$$\int_{-\infty}^{\infty} \psi(x, y)G(x)dx = C^*(\lambda; G)g(y) \qquad \text{a.e. in } B_{\lambda,\varepsilon}. \tag{3.1}$$

For $\phi(x/y) \in N_f$ we have

$$\int_{-\infty}^{\infty} \phi\left(-\frac{x}{y}\right) G(x)dx = 0 \qquad \text{for every } y > 0. \tag{3.2}$$

PROOF. Let $\psi \in N_f$ and $K(x, y) = yH(y)G(x)$ where $H \in L_1(\mathbf{R}^+)$. Then $K \in L_1(G)$ and we have

$$\int_G g(\mu y)e^{-i\mu x}G(x)H(y)\frac{1}{y}\,dydx = \hat{G}(\mu)\int_0^\infty g(\mu y)H(y)\frac{1}{y}\,dy.$$

Suppose that $\overline{H(y)}$ is supported by $B_{\lambda,\varepsilon}$ and orthogonal to $d_\lambda y^{i\lambda}\,|_{B_{\lambda,\varepsilon}}$.

Hence, we have

$$\hat{G}(\mu)\int_0^\infty g(\mu y)H(y)\frac{1}{y}\,dy = \hat{G}(\mu)\mu^{i\lambda}d_\lambda\int_{B_{\lambda,\varepsilon}} y^{i\lambda}H(y)\frac{1}{y}\,dy = 0$$

for each $\mu > 0$.

Therefore, the function $\overline{K(x, y)}$ is orthogonal to ψ which yields

$$\int_{B_{\lambda,\varepsilon}}\left[\int_{-\infty}^{\infty}\psi(x, y)G(x)dx\right]H(y)\frac{1}{y}\,dy = 0.$$

Hence, there exists a constant $C^*(\lambda; G)$ such that

$$\int_{-\infty}^{\infty}\psi(x, y)G(x)dx = C^*(\lambda; G)d_\lambda y^{i\lambda} \qquad \text{a.e. in } B_{\lambda,\varepsilon}.$$

Let $\phi(x/y) \in N_f$ and suppose that there exists a positive real y_0 such that

$\int_{-\infty}^{\infty}\phi(-x/y_0)G(x)dx \neq 0$. From the continuity of the function $s(y)=\int_{-\infty}^{\infty}\phi(-x/y)G(x)dx$ we deduce that there exists $H \in L_1(\mathbf{R}^+)$ supported by a neighbourhood of y_0 such that

$$\int_0^\infty \left[\int_{-\infty}^\infty \phi\left(-\frac{x}{y}\right) G(x)dx\right] H(y)\frac{1}{y}\,dy$$
$$=\int_0^\infty \left[\int_{-\infty}^\infty \phi\left(\frac{x}{y}\right) G(-x)dx\right] H(y)\frac{1}{y}\,dy \neq 0.$$

However, for every $\mu > 0$ we have

$$\int_G g(\mu y)e^{-i\mu x}G(-x)H(y)\frac{1}{y}\,dydx = \hat{G}(-\mu)\int_0^\infty g(\mu y)H(y)\frac{1}{y}\,dy = 0$$

contradicting the assumption that $\phi \in N_f$.

Some partial information about $\mathrm{Sp}(\phi)$ is provided by the following lemma.

LEMMA 3.2. *If $\phi(x/y) \in N_f$, $\phi \neq 0$, then* $\mathrm{Sp}(\phi) \subset (0,\infty)$.

PROOF. Suppose that $s \in \mathrm{Sp}(\phi)$, $s<0$. Then $1 \in \mathrm{Sp}(\phi(x/s))$.

Consequently, there exists $G \in S(\mathbf{R})$ with $\mathrm{Supp}\,\hat{G} \subseteq [1-\varepsilon, 1+\varepsilon]$, $0<\varepsilon<1$, such that $\int_{-\infty}^{\infty}\phi(x/y)G(x)dx \neq 0$ contradicting (3.2).

Suppose that $0 \in \mathrm{Sp}(\phi)$. Then for every $0<\varepsilon<1$ there exists $G \in S(\mathbf{R})$ with $\mathrm{Supp}\,\hat{G} \subseteq [-\varepsilon, \varepsilon]$ such that $\int_{-\infty}^{\infty}\phi(x)G(x)dx \neq 0$. Hence, there exists $H \in L_1(\mathbf{R}^+)$ supported by $[1-\delta, \delta]$, $0<\delta<1$, such that

$$\int_G \phi\left(\frac{x}{y}\right) G(x)H(y)\frac{1}{y}\,dydx \neq 0.$$

On the other hand, we have

$$\int_G g(\mu y)e^{-i\mu x}G(x)H(y)\frac{1}{y}\,dydx = \hat{G}(\mu)\int_{1-\delta}^{\delta} g(\mu y)H(y)\frac{1}{y}\,dy = 0$$

for every $\mu > 0$. Contradiction.

The following lemma supplies more information about $\hat{\phi}$ for $\phi(x/y) \in N_f$.

LEMMA 3.3. *Suppose that $\phi(x/y) \in N_f$, $\phi \neq 0$. Then for every ε, $0<\varepsilon<1$, there exists $G \in S(\mathbf{R})$ with* $\mathrm{Supp}\,\hat{G} \subseteq [1-\varepsilon, 1+\varepsilon]$ *such that the function $C^*(\lambda; G)$ on Γ has an extension to a continuous function that vanishes nowhere on* $\mathbf{R}$. *Furthermore, for every $G \in S(\mathbf{R})$ with* $\mathrm{Supp}\,\hat{G} \subseteq [1-\varepsilon, 1+\varepsilon]$ *there exists a constant d such that*

$$C^*(\lambda; G) = d\widehat{\hat{G}(e^{-x})}(\lambda).$$

Let

$$P(\lambda\,;G)=\hat{G}(e^{-x})(\lambda). \tag{3.3}$$

Obviously, we have

$$P(\lambda\,;G)=\int_0^\infty \frac{\hat{G}(w)}{w^{1-i\lambda}}\,dw.$$

For the proof of Lemma 3.3, we need the following two lemmas.

LEMMA 3.4. *We have $C^*(\lambda\,;G)=0$ if and only if*

$$P(\lambda\,;G)=0.$$

PROOF. Suppose that $P(\lambda\,;G)=0$ for some $\lambda\in\Gamma$ such that $B_{\lambda,\varepsilon}$ is defined. Let $y_0\in B_{\lambda,\varepsilon}$. Hence we have

$$\int_{-\infty}^{\infty}\phi\left(\frac{x}{y_0}\right)G(x)dx=C^*(\lambda\,;G)d_\lambda y_0^{i\lambda}.$$

Let y_1 and y_2 be positive reals which satisfy

$$\frac{1+\varepsilon}{y_2}<\frac{1-\varepsilon}{y_1}\,.$$

Let $M\in S(\mathbf{R})$ with

$$\operatorname{Supp}\hat{M}\subset\left(\frac{y_0(1+\varepsilon)}{y_2},\frac{y_0(1-\varepsilon)}{y_1}\right)\quad\text{such that}\quad\int\frac{\hat{M}(z)}{z}\,dz\neq 0. \tag{3.4}$$

Let K be defined as

$$K(x,y)=\begin{cases}\displaystyle\int_{-\infty}^{\infty}G\left(\frac{y_0(x-\xi)}{y}\right)M(\xi)d\xi & y_1<y<y_2,\\[2ex] 0 & \text{elsewhere.}\end{cases}$$

We notice that for every positive integer n, there exists $C_n>0$ such that

$$|K(x,y)|\leqq\frac{C_n}{1+|x|^n}\,,\qquad x,y\in\mathbf{R}.$$

Hence $K\in L_1(G)$. The function $\overline{K(x,y)}$ is orthogonal to $g(\mu y)e^{-i\mu x}$ for each $\mu>0$, as seen in the following:

$$\int_{y_1}^{y_2}\int_{-\infty}^{\infty}g(\mu y)e^{-i\mu x}K(x,y)\frac{1}{y^2}\,dxdy$$

$$= \int_{y_1}^{y_2} \frac{1}{y^2} g(\mu y) \left[\widehat{\int_{-\infty}^{\infty}} \left[\int_{-\infty}^{\infty} G\left(\frac{y_0(x-\xi)}{y}\right) M(\xi) d\xi \right] e^{-i\mu x} dx \right] dy$$

$$= \int_{y_1}^{y_2} \frac{1}{y^2} g(\mu y) \widehat{G\left(\frac{y_0 x}{y}\right)} (\mu) \hat{M}(\mu) dy \tag{3.5}$$

$$= \frac{1}{y_0} \hat{M}(\mu) \int_{y_1}^{y_2} \hat{G}\left(\frac{\mu y}{y_0}\right) g(\mu y) \frac{1}{y} dy$$

$$= \frac{1}{y_0} \hat{M}(\mu) \int_{\mu y_1/y_0}^{\mu y_2/y_0} \frac{\hat{G}(z)}{z} g(y_0 z) dz$$

$$= \frac{d_\lambda}{y_0^{1-i\lambda}} \hat{M}(\mu) \int_{\mu y_1/y_0}^{\mu y_2/y_0} \frac{\hat{G}(z)}{z^{1-i\lambda}} dz.$$

For μ in $(y_0(1+\varepsilon)/y_2, y_0(1-\varepsilon)/y_1)$ we have

$$\operatorname{Supp} \hat{G} \subseteq [1-\varepsilon, 1+\varepsilon] \subset \left(\frac{\mu y_1}{y_0}, \frac{\mu y_2}{y_0}\right)$$

and, consequently,

$$\int_{\mu y_1/y_0}^{\mu y_2/y_0} \frac{\hat{G}(z)}{z^{1-i\lambda}} dz = P(\lambda; G) = 0.$$

Thus, by (3.4), (3.5) is equal to zero for each $\mu > 0$. As $\phi(x/y) \in M_f$, the function $\overline{K(x,y)}$ must therefore be orthogonal to $\phi(x/y)$. That is,

$$\int_{y_1}^{y_2} \int_{-\infty}^{\infty} \phi\left(\frac{x}{y}\right) K(x,y) \frac{1}{y^2} dx dy = 0. \tag{3.6}$$

On the other hand, we have

$$\int_{y_1}^{y_2} \int_{-\infty}^{\infty} \phi\left(\frac{x}{y}\right) K(x,y) \frac{1}{y^2} dx dy$$

$$= \int_{y_1}^{y_2} \left[\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} G\left(\frac{y_0 x}{y} - \frac{y_0 \xi}{y}\right) M(\xi) d\xi \right] \phi\left(\frac{x}{y}\right) dx \right] \frac{1}{y^2} dy$$

$$= \frac{1}{y_0} \int_{y_1}^{y_2} \left[\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} G\left(z - \frac{y_0 \xi}{y}\right) M(\xi) d\xi \right] \phi\left(\frac{z}{y_0}\right) dz \right] \frac{1}{y} dy.$$

If we represent G as $G(w) = (1/2\pi) \int_{1-\varepsilon}^{1+\varepsilon} \hat{G}(s) e^{iws} ds$ we obtain:

$$= \frac{1}{2\pi y_0} \int_{y_1}^{y_2} \left[\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \left[\int_{1-\varepsilon}^{1+\varepsilon} \hat{G}(s) e^{is(z - y_0 \xi/y)} ds \right] M(\xi) d\xi \right] \phi\left(\frac{z}{y_0}\right) dz \right] \frac{1}{y} dy$$

$$= \frac{1}{2\pi y_0} \int_{y_1}^{y_2} \left[\int_{-\infty}^{\infty} \left[\int_{1-\varepsilon}^{1+\varepsilon} \hat{M}\left(\frac{y_0 s}{y}\right) \hat{G}(s) e^{isz} ds \right] \phi\left(\frac{z}{y_0}\right) dz \right] \frac{1}{y} dy.$$

By applying Fubini's theorem twice we obtain

$$
\begin{aligned}
&= \frac{1}{2\pi y_0}\int_{-\infty}^{\infty}\left[\int_{1-\varepsilon}^{1+\varepsilon}\left[\int_{y_1}^{y_2}\frac{1}{y}\hat{M}\left(\frac{y_0 s}{y}\right)dy\right]\hat{G}(s)e^{isz}ds\right]\phi\left(\frac{z}{y_0}\right)dz \\
(3.7)\quad &= \frac{1}{2\pi y_0}\int_{-\infty}^{\infty}\left[\int_{1-\varepsilon}^{1+\varepsilon}\left[\int_{y_0 s/y_2}^{y_0 s/y_1}\frac{\hat{M}(t)}{t}dt\right]\hat{G}(s)e^{isz}ds\right]\phi\left(\frac{z}{y_0}\right)dz.
\end{aligned}
$$

By (3.4) we have

$$\int_{y_0 s/y_2}^{y_0 s/y_1}\frac{\hat{M}(t)}{t}dt = \int \frac{\hat{M}(t)}{t}dt \qquad \text{for every } s \text{ in } [1-\varepsilon, 1+\varepsilon].$$

Thus, (3.7) is equal to

$$\frac{1}{y_0}\int \frac{\hat{M}(t)}{t}dt \cdot \int_{-\infty}^{\infty} G(z)\phi\left(\frac{z}{y_0}\right)dz = C^*(\lambda; G)\frac{d_\lambda}{y_0^{1-i\lambda}}\int \frac{\hat{M}(t)}{t}dt$$

which yields, by (3.6), that $C^*(\lambda; G) = 0$ and the "if" part of the lemma is proved.

To prove the "only if" part suppose that $C^*(\lambda; G) = 0$ for some $\lambda \in \Gamma$ and that $P(\lambda; G) \neq 0$.

By Wiener's theorem there exists $s \in \mathbf{R}$, $s \in \mathrm{Sp}(\phi)$. From Lemma 3.2 we deduce that $s > 0$. Let $G_1 \in S(\mathbf{R})$ with $\mathrm{Supp}\, G_1 \subseteq [1-\varepsilon, 1+\varepsilon]$ such that $\int_{-\infty}^{\infty}\phi(x/s)G_1(x)dx = C_1 \neq 0$.

Let y_1, y_2 be positive reals such that $(1+\varepsilon)/y_2 < (1-\varepsilon)/y_1$ and let $M \in S(\mathbf{R})$ where $\mathrm{Supp}\,\hat{M} \subset (s(1+\varepsilon)/y_2, s(1-\varepsilon)/y_1)$. Let $K_1 \in L_1(G)$ be defined as

$$K_1(x,y) = \begin{cases} \int_{-\infty}^{\infty} G_1\left(\frac{s(x-\xi)}{y}\right)M(\xi)d\xi & y_1 \leqq y \leqq y_2, \\ 0 & \text{elsewhere.}\end{cases}$$

As in the "if" part of the proof we obtain

$$\int_G G(\mu y)e^{-i\mu x}K_1(x,y)\frac{1}{y^2}dydx = C_2\hat{M}(\mu)$$

where

$$C_2 = \frac{1}{s}\int_{1-\varepsilon}^{1+\varepsilon}\frac{\hat{G}(z)}{z}g(sz)dz.$$

Let $y_0 \in B_{\lambda,\varepsilon}$, y_3, y_4 be positive reals satisfying $y_4 > y_3$, $y_0/y_3 = s/y_1$ and $y_0/y_4 = s/y_2$.

Finally, let $K_2 \in L_1(G)$ be defined as

$$K_2(x,y) = \begin{cases} \int_{-\infty}^{\infty} G\left(\frac{y_0(x-\xi)}{y}\right) M(\xi)d\xi & y_3 < y < y_4, \\ 0 & \text{elsewhere.} \end{cases}$$

Then we have

$$\int_G g(\mu y)e^{-i\mu x}K_2(x,y)\frac{1}{y^2}\,dydx = \hat{M}(\mu)\frac{d_\lambda}{y_0^{1-i\lambda}}P(\lambda;G) = C_3\hat{M}(\mu).$$

Consequently, the function $\overline{K(x,y)} = \overline{C_3}\,\overline{K_1(x,y)} - \overline{C_2}\,\overline{K_2(x,y)}$ belongs to $L_1(G)$ and is orthogonal to $g(\mu y)e^{-i\mu x}$ for all $\mu > 0$.

However, we have

$$\int_G \phi\left(\frac{x}{y}\right) K(x,y)\frac{1}{y^2}\,dydx = \left[C_1C_3\frac{1}{s} - C^*(\lambda;G)\frac{C_2 d_\lambda}{y_0^{1-i\lambda}}\right]\int \frac{\hat{M}(t)}{t}\,dt$$

$$= C_1C_3\frac{1}{s}\int \frac{\hat{M}(t)}{t}\,dt \neq 0$$

contradicting the assumption that $\phi(x/y) \in N_f$. This completes the proof of Lemma 3.4.

LEMMA 3.5. *We have*

$$C^*(\lambda_1;G)P(\lambda_2;G) = C^*(\lambda_2;G)P(\lambda_1;G) \qquad \textit{for } \lambda_1, \lambda_2 \in \Gamma.$$

PROOF. Suppose that $P(\lambda_i;G) \neq 0$, $i = 1,2$. Let y_0, y_0^*, y_i $(i = 1,2,3,4)$ be positive reals satisfying

$$\frac{1+\varepsilon}{y_2} < \frac{1-\varepsilon}{y_1}, \quad \frac{y_0^*}{y_3} = \frac{y_0}{y_1}, \quad \frac{y_0^*}{y_4} = \frac{y_0}{y_2}, \qquad y_0 \in B_{\lambda_1,\varepsilon} \quad \text{and} \quad y_0^* \in B_{\lambda_2,\varepsilon}.$$

Let $M \in S(\mathbf{R})$ with

$$\operatorname{Supp}\hat{M} \subset \left(\frac{y_0(1+\varepsilon)}{y_2}, \frac{y_0(1-\varepsilon)}{y_1}\right) \qquad \text{such that} \int \frac{\hat{M}(t)}{t}\,dt \neq 0.$$

Let K_i $(i = 1,2)$ be functions in $L_1(G)$ defined as:

$$K_1(x,y) = \begin{cases} b\int_{-\infty}^{\infty} G\left(\frac{y_0(x-\xi)}{y}\right) M(\xi)d\xi & y_1 < y < y_2 \\ 0 & \text{elsewhere} \end{cases}$$

where

$$b = \frac{P(\lambda_2; G)}{P(\lambda_1; G)} \frac{d_{\lambda_2}}{d_{\lambda_1}} \frac{y_0^{1-i\lambda_1}}{y_0^{*1-i\lambda_2}}, \tag{3.8}$$

and

$$K_2(x, y) = \begin{cases} \int_{-\infty}^{\infty} G\left(\frac{y_0^*(x-\xi)}{y}\right) M(\xi) d\xi & y_3 < y < y_4, \\ 0 & \text{elsewhere.} \end{cases}$$

Consider the function $K(x, y) = K_1(x, y) - K_2(x, y)$. We have, by (3.8),

$$\int_G g(\mu y) e^{-i\mu x} K(x, y) \frac{1}{y^2} dx dx = \left[\frac{bd_{\lambda_1}}{y_0^{1-i\lambda_1}} P(\lambda_1; G) - \frac{d_{\lambda_2}}{y_0^{*1-i\lambda_2}} P(\lambda_2; G)\right] \hat{M}(\mu) = 0$$

for all $\mu > 0$.

As $\phi(x/y) \in N_f$, we conclude that

$$\int_G \phi\left(\frac{x}{y}\right) K(x, y) \frac{1}{y^2} dy dx = \left[b \frac{C^*(\lambda_1; G) d_{\lambda_1}}{y_0^{1-i\lambda_1}} - \frac{C^*(\lambda_2; G) d_{\lambda_2}}{y_0^{*1-i\lambda_2}}\right] \int \frac{\hat{M}(t)}{t} dt = 0.$$

From (3.8) we deduce

$$\frac{C^*(\lambda_1; G)}{P(\lambda_1; G)} = \frac{C^*(\lambda_2; G)}{P(\lambda_2; G)}$$

and the result follows.

Proof of Lemma 3.3. Let $G \in S(\mathbf{R})$, $G \neq 0$ with $\operatorname{Supp} \hat{G} \subseteq [1-\varepsilon, 1+\varepsilon]$ for some ε, $0 < \varepsilon < 1$. Let $P(\lambda; G) = \widehat{G(e^{-x})}(\lambda)$. If $P(\lambda; G) = 0$ for each $\lambda \in \Gamma$ then from Lemma 3.4 we deduce $C^*(\lambda; G) = 0$ for $\lambda \in \Gamma$ which yields Lemma 3.3 for $d = 0$.

Suppose that $P(\lambda_1; G) \neq 0$ for some $\lambda_1 \in \Gamma$. By Lemma 3.4 we have $C^*(\lambda_1; G) \neq 0$. Let $d = C^*(\lambda_1; G)/P(\lambda_1; G)$. Hence, by Lemma 3.5 we deduce that $C^*(\lambda; G) = dP(\lambda; G)$ for every $\lambda \in \Gamma$.

For $0 < \varepsilon < 1$ we construct a function $G \in S(\mathbf{R})$ with $\operatorname{Supp} \hat{G} \subseteq [1-\varepsilon, 1+\varepsilon]$ such that $P(\lambda; G) > 0$ for every real λ.

Let $\psi \in S(\mathbf{R})$ with $\operatorname{Supp} \hat{\psi} \subseteq [-\delta/2, \delta/2]$ where $\delta > 0$ satisfies $[e^{-\delta}, e^{\delta}] \subset [1-\varepsilon, 1+\varepsilon]$. By the analyticity of $\psi(z)$, $z \in \mathbf{C}$, there exists $z_0 \in \mathbf{C}$ such that $\alpha(x) = \psi(x + i \operatorname{Im} z_0)$ vanishes nowhere on $\mathbf{R}$ and $\operatorname{Supp} \hat{\alpha} = \operatorname{Supp} \hat{\psi}$. Let $P_1(\xi) = (\widehat{\hat{\alpha}(x) * \hat{\alpha}(-x)})(\xi)$. Then $P_1 \in S(\mathbf{R})$, $P_1(\lambda) > 0$ for each $\lambda \in \mathbf{R}$ and $\operatorname{Supp} \hat{P}_1 \subseteq [-\delta, \delta]$.

Let

$$\hat{G}(x) = \begin{cases} P_1(-\ln x) & x \in [e^{-\delta}, e^{\delta}], \\ 0 & \text{elsewhere.} \end{cases}$$

Then $\operatorname{Supp}\hat{G} \subseteq [1-\varepsilon, 1+\varepsilon]$, and by (3.3), $P_1(\lambda) = P(\lambda; G)$.
This completes the proof of the lemma.

REMARK. By definition the set Γ is dense in $\mathbf{R}$. Therefore, if $G \neq 0$ the function $P(\lambda; G)$ is not identically zero on Γ. Consequently, the constant $d \neq 0$.

It follows by Lemma 3.3 that if $\phi(x/y) \in N_f$, $\phi \neq 0$, then for every $G \in S(\mathbf{R})$ with $\operatorname{Supp}\hat{G} \subseteq [1-\varepsilon, 1+\varepsilon]$, $0 < \varepsilon < 1$, the function $Q(y) = \int_{-\infty}^{\infty} \phi(x/y)G(x)dx$ is determined by $g(y)$ on the intervals $B_{\lambda,\varepsilon}$, $\lambda \in \Gamma$.

The following lemma implies that $(1/y)Q(y)$ must possess an additional property which cannot be possessed by $g(y)$.

LEMMA 3.6. *Let $\phi \in L_\infty(\mathbf{R})$, $\phi \neq 0$ and $G \in S(\mathbf{R})$ such that $\hat{G}(0) = 0$. Then there exists a constant $C > 0$ such that*

$$\left| \int_{1<|y|<N} \left[\int_{-\infty}^{\infty} \phi(x)G(xy)dx \right] dy \right| < C$$

for any $N > 1$.

PROOF. Let $K(y) = \int_{-\infty}^{\infty} \phi(x)G(xy)dx$. The function $R(x,y) = \phi(x)G(xy)$ is integrable in the strip $\{(x,y): 1 < |y| < N,\ x \in \mathbf{R}\}$.

Thus, by Fubini's theorem, we have

$$\int_{1<|y|<N} K(y)dy = \int_{-\infty}^{\infty} \left[\int_{1<|y|<N} G(xy)dy \right] \phi(x)dx$$

and

$$\int_{1<|y|<N} G(xy)dy = \frac{1}{|x|} \left[\int_{-Nx}^{Nx} G(z)dz - \int_{-x}^{x} G(z)dz \right]$$

for any $x \neq 0$.

Let $M(w) = \int_{-\infty}^{w} G(x)dx$. As $\hat{G}(0) = 0$ it follows that $M \in S(\mathbf{R})$ and hence we have:

$$\begin{aligned} B_N(x) &= \int_{1<|y|<N} G(xy)dy \\ &= \frac{1}{|x|}(M(Nx) - M(-Nx)) + \frac{1}{|x|}(M(x) - M(-x)) \end{aligned} \tag{3.9}$$

for any $N > 1$ and $x \neq 0$.

Let $T(x) = (1/|x|)(M(x) - M(-x))$ for $x \neq 0$.

By (3.9) we obtain $B_N(x) = NT(Nx) + T(x)$.

$T(x) \in L_1(\mathbf{R})$ as seen in the following:

$$\int_{-\infty}^{\infty} |T(x)|\,dx = \int_{|x|\leq 1} \frac{1}{x}|M(x) - M(-x)|\,dx + \int_{|x|>1} \frac{1}{|x|}|M(x) - M(-x)|\,dx$$

$$< 4 \max_{|x|\leq 1} |M'(x)| + 2\|M\|_{L_1} = C_1.$$

Hence,

$$\|B_N(x)\|_{L_1} \leqq N\|T(Nx)\|_{L_1} + \|T\|_{L_1} = 2\|T\|_{L_1} < 2C_1$$

for any $N > 1$.

Finally, we have

$$\left| \int_{1<|y|<N} K(y)\,dy \right| \leqq \|B_N(x)\|_{L_1} \|\phi\|_{L_\infty} < C \qquad \text{for any } N > 1$$

as required.

Proof of the Theorem. Suppose that N_f contains a function $y^{i\theta}\phi(x/y)$ where $\phi \in L_\infty(\mathbf{R})$, $\phi \neq 0$, $\theta \in \mathbf{R}$. Obviously, $N_{y^{-i\theta}f} = y^{-i\theta}N_f$ and $N_{y^{-i\theta}f}$ contains the function $\phi(x/y)$. The function $g_\theta(y) = y^{-i\theta}g(y)$ is defined as $d_\lambda y^{i(\lambda-\theta)}$ on the intervals $(\underline{y}, \bar{y}_\lambda]$. By Lemma 3.3, there exists $G \in S(\mathbf{R})$ with $\operatorname{Supp}\hat{G} \subseteq [1-\varepsilon, 1+\varepsilon]$, $0 < \varepsilon < 1$, such that $P(\lambda; G)$, as defined by (3.3), is positive on $\mathbf{R}$, and $C^*(\lambda; G) = dP(\lambda; G)$ for $\lambda \in \Gamma$ and $d \neq 0$.

From Lemma 3.1 we have

$$\int_{-\infty}^{\infty} \phi\left(\frac{x}{y}\right) G(x)\,dx = C^*(\lambda; G)\,d_\lambda y^{i(\lambda-\theta)} \qquad \text{for every } y \text{ in } B_{\lambda,\varepsilon}$$

and

$$\int_{-\infty}^{\infty} \phi\left(\frac{x}{y}\right) G(x)\,dx = 0 \qquad \text{for any } y < 0.$$

Let $K(y) = \int_{-\infty}^{\infty} \phi(x)G(xy)\,dx$. Then, by Lemma 3.6 there exists a positive real C such that

$$\left| \int_{y_1}^{y_2} K(y)\,dy \right| < C \tag{3.10}$$

for every $y_1, y_2 > 1$.

The function $P(w; G)$ is positive and continuous on $\mathbf{R}$. Consequently, there exists $\delta > 0$ such that

$$P(w; G) > \frac{1}{2} P(\theta; G) \qquad \text{for } |w - \theta| < \delta.$$

By the definition of $g(y)$ in the theorem, it follows that for every $M > 0$ there exists $\lambda_0 \in \Gamma$ such that $|\lambda_0 - \theta| < \delta$ and

$$\frac{\bar{y}_{\lambda_0}}{\underline{y}_{\lambda_0}} > e^M.$$

For any $\lambda > 0$ we have

$$\left| \int_{y_1}^{y_2} \frac{1}{y^{1-i\alpha}} dy \right| = \left| \int_{\lambda y_1}^{\lambda y_2} \frac{1}{y^{1-i\alpha}} dy \right|, \qquad \alpha \in \mathbf{R}$$

and

$$\lim_{\alpha \to 0} \int_{y_1}^{y_2} \frac{1}{y^{1-i\alpha}} dy = \ln \frac{y_2}{y_1}.$$

Consequently, there exist $\lambda_0 \in \Gamma$, $y_1, y_2 \in B_{\lambda_0,\varepsilon}$ satisfying the following conditions:

(3.11) $$|\lambda_0 - \theta| < \delta,$$

(3.12) $$\frac{y_2}{y_1} > \exp\left(\frac{4C}{|d| P(\theta; G)}\right),$$

(3.13) $$\left| \int_{y_1}^{y_2} \frac{1}{y^{1-i(\lambda_0-\theta)}} dy \right| > \frac{1}{2} \ln \frac{y_2}{y_1}.$$

Hence, we conclude

$$\left| \int_{y_1}^{y_2} K(y) dy \right| = |C^*(\lambda_0; G)| \left| \int_{y_1}^{y_2} \frac{1}{y^{1-i(\lambda_0-\theta)}} dy \right|$$

$$> \frac{1}{4} |d| P(\theta; G) \ln \frac{y_2}{y_1} > C$$

Contradicting (3.10).

This completes the proof of the theorem.

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